The Four Meanings of “Accurate to 3 Places”
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INTRODUCTION

If someone were to tell you to recite \( \pi \) accurately to three places, you would probably say “3.141” since those are the first three decimal places of \( \pi \). But if you were asked, “What is \( \pi \) accurate to 3 places?” you would put your HP calculator in FIX 3 mode, press \( \pi \), and reply, “3.142”. Two different answers, but both are correct.

Even further, suppose somebody were to ask, “What fraction does the HP 50g return when the input is \( \pi \), the display is in FIX 3 mode, and the \( \rightarrow \mathbf{Q} \) function is executed?” The answer is \( \frac{333}{106} \). But exactly what effect does FIX 3 have on \( \rightarrow \mathbf{Q} \)? The answer is more complicated than the simple examples in the previous paragraph. FIX 3 tells \( \rightarrow \mathbf{Q} \) that the answer must not differ from the input by more than \( 10^{-3} \). In other words, \( \rightarrow \mathbf{Q} \) looks for \( \text{input} - \text{output} \leq 0.001 \). Notice the three digits after the decimal point. That’s what FIX 3 tells \( \rightarrow \mathbf{Q} \) to look for. This is yet a third meaning of “accurate to 3 places”.

A fourth meaning must be addressed: What answer does my teacher’s calculator give? No matter how good my calculator is objectively, it is worse than useless if it gives answers that differ from the teacher’s calculator, since that is the norm used for grading, especially if the teacher is unfamiliar with other calculator models.

Thus we have four radically different meanings of “accurate to three places”, namely

1. Truncated to three decimal places.
2. Rounded to three decimal places.
3. Differing from the input by \( \leq 0.001 \)
4. Whatever the teacher’s calculator says.

In this paper, we will examine an example for which all four meanings have different values, namely, \( \sqrt{84} \).

CONTINUED FRACTIONS

To accomplish our task, we must break down \( \sqrt{84} \) into its equivalent “continued fraction.”

Consider the following expression of nested reciprocals:

\[
1 + \frac{1}{2 + \frac{1}{3 + \frac{1}{4}}}.
\]

Critters that look like this are called “continued fractions.” Converting continued fractions to simple fractions is easy, working from the bottom up, as you can see below. Be sure to follow each step.
To avoid this cumbersome notation, continued fractions are usually written as a list containing the leading integer and then the denominators. For example, the above continued fraction is written as \([1, 2, 3, 4]\). The numbers in the list are called the “partial quotients” of the continued fraction.

Everybody knows that the square roots of non-square integers are non-repeating, non-terminating decimal numbers, right? Amazingly, square roots are all repeating continued fractions! The partial quotients can be seen to fit a repeating pattern. For example, \(\sqrt{84} = [9, 6, 18, 6, 18, 6, 18, \ldots]\) repeating forever.

Some other surprising continued fractions are:

The golden ratio \(\left(\frac{\sqrt{5} + 1}{2}\right) = [1, 1, 1, 1, 1, \ldots]\). This is the simplest possible continued fraction.

\(e = [2, 1, 2, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, 10, 1, 1, 12, \ldots]\)

\(\tan(1 \text{ radian}) = [1, 1, 1, 3, 1, 5, 1, 7, 1, 9, 1, 11, 1, 13, \ldots]\)

Unfortunately, not all irrational numbers have continued fractions with repeating partial quotients. For example, \(\pi = [3, 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, 2, 1, 1, 2, 2, 2, 2, 1, 84, 2, 1, 1, 15, 3, \ldots]\). It never terminates, but it never repeats.

**ITERATIVE FRACTION GENERATION**

There is a truly marvelous method for generating approximate fractions for any number. As an example, let’s find the best fractions that approximate \(\pi\).

First you make a list of the continued fraction of \(\pi\), stopping at the first large number (let’s stop at 292). Then you create a table with three rows. Fill in the top row with the list of partial quotients (see the red numbers above) and fill in leading 0’s and 1’s like this:

<table>
<thead>
<tr>
<th></th>
<th>3</th>
<th>7</th>
<th>15</th>
<th>1</th>
<th>292</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The bottom two rows represent the fractions that approximate \(\pi\), beginning with 0/1 (zero) and 1/0 (infinity). As we proceed, the fractions will get closer and closer to \(\pi\) with amazing rapidity.
Fill in the boxes with this pattern: Use the first and second row to get $3\times1+0 = 3$ and fill it in here:

\[
\begin{array}{cccccc}
3 & 7 & 15 & 1 & 292 \\
0 & 1 & 3 & & & \\
1 & 0 & & & \\
\end{array}
\]

Now do the same thing with the first and third row: $3\times0+1 = 1$ and fill it in there:

\[
\begin{array}{cccccc}
3 & 7 & 15 & 1 & 292 \\
0 & 1 & 3 & & & \\
1 & 0 & 1 & & & \\
\end{array}
\]

Thus our first fraction approximating $\pi$ is $3/1$. Not very impressive, but it gets better quickly. Following the same pattern as before, do these two calculations and fill them in: $7\times3+1 = 22$ and $7\times1+0 = 7$:

\[
\begin{array}{cccccc}
3 & 7 & 15 & 1 & 292 \\
0 & 1 & 3 & 22 & & \\
1 & 0 & 1 & 7 & & \\
\end{array}
\]

Thus $\pi$ is approximately $22/7$, the approximation they taught us in school. Do the next column: $15\times22+3 = 333$ and $15\times7+1 = 106$:

\[
\begin{array}{cccccc}
3 & 7 & 15 & 1 & 292 \\
0 & 1 & 3 & 22 & 333 & \\
1 & 0 & 1 & 7 & 106 & \\
\end{array}
\]

This is surprising! $333/106$ is a better approximation than $22/7$, but nobody ever mentions it! Continue the process: $1\times333+22 = 355$ and $1\times107+7 = 113$:

\[
\begin{array}{cccccc}
3 & 7 & 15 & 1 & 292 \\
0 & 1 & 3 & 22 & 333 & 355 \\
1 & 0 & 1 & 7 & 106 & 113 \\
\end{array}
\]

Ah yes, we’ve all heard of $355/113$, which is even better than $333/106$. Now do the final column: $292\times355+33 = 103993$ and $292\times113+106 = 33102$:

\[
\begin{array}{cccccc}
3 & 7 & 15 & 1 & 292 \\
0 & 1 & 3 & 22 & 333 & 355 & 103993 \\
1 & 0 & 1 & 7 & 106 & 113 & 33102 \\
\end{array}
\]

Thus our final approximation is $103993/33102$. 
Table 1 shows the first 14 fractions that approximate √84. Which one best approximates √84 accurate to three places?

(1) If “accurate to three places” means “differing from √84 by less than 0.001,” then we look down the last column for the first “error” starting with three zeros. We find it in row 7. Therefore, the correct answer is 614/67.

(2) If “accurate to three places” means “displaying the same in FIX 3 mode,” then we look down the “Approx.” column for the first value that rounds to 9.165. We find it in row 9. Therefore, the correct answer is 724/79.
(3) If “accurate to three places” means “having exactly the same three digits, truncated,” then we look down the “Approx.” column for the first value that begins with exactly the digits 9.165. We find it in row 13. Therefore the correct answer is 944/103.

(4) If “accurate to three places” means “Whatever the teacher gets,” then we look down the “Continued Fraction” column for the first whole subset of partial quotients below the answer obtained in (1) above. (!!!) This is the answer obtained by every calculator on the planet (except for the HP-33s and HP-35s, and the HP 49/50 running the PDQ algorithm, none of which are used by teachers). We find this entry in row 14, which has the complete subset of partial quotients [9, 6, 18]. Therefore the correct answer is 999/109.

Note: If the teacher actually has an HP-33s or HP-35s, or an HP 49/50 with PDQ in it, then they will understand the above already and will give full marks to any student who obtains any of these “correct” answers.

IMPLEMENTATION

Calculator designers who decide to include a “fraction button” are therefore faced with a difficult choice (unless they are ignorant of it). Which of these four meanings of “accurate to three places” should be implemented in their calculator? HP’s RPL models feature a function called \( \rightarrow Q \) (“to Quotient”). It turns a decimal number into a fraction whose accuracy is controlled by the display’s FIX setting, as we saw in the second paragraph of this paper. In other words, it uses definition (1) above. So does the PPC ROM’s “DF” (“Decimal to Fraction”) program.

The September 2011 issue of HP Solve features an HP-15C program that converts decimals to fractions using definition (2) above. The FIX setting is used to round the input and the output until they are the same. No other calculator or program does this, to my knowledge.

Ordinary people use definition (3) above. That’s why nobody implements it.

The HP 32SII uses definition (4) above. So do all non-HP calculators that have a fraction button with user-controllable accuracy.

My suggestion to calculator designers is to follow definition (2) and use the FIX setting to control the accuracy. When the rounded input equals the rounded output, stop. The amount of required math is roughly the same as the other definitions above, and is easily implemented.

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